MATH 2271

Chapter 11.2

Fourier Series

Let p > 0 be a fixed number and f(x) be a periodic function with a period 2p, defined on (-p, p). The Fourier series of f(x) is a way of expanding the function f(x) into an infinite series involving sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right).$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx. \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx. \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx.$$

Note: P is half the period!

From a practical point of view, it suffices to calculate the integrals above and plug them into the big formula. Couple of other things to know:

- $\sin(+\text{-}n\pi) = 0$
- $\cos(+\text{-}n\pi) = (-1)^n$
- $\cos(-n) = \cos(n)$
- $\sin(-n) = -\sin(n)$

Some useful trig identities

- $sin(a)sin(b) = ½[cos(a - b) - cos(a + b)]$
- $cos(a)cos(b) = ½[cos(a - b) + cos(a + b)]$
- $sin(a)cos(b) = ½[sin(a + b) + sin(a - b)]$
- $sin(a + b) = sin(a)cos(b) + cos(a)sin(b)$
- $cos(a + b) = cos(a)cos(b) \mp sin(a)\,sin(b)$

Fourier Series Convergence

Let f and f' be piecewise continuous on the interval $[-p, p]$. Then for all x in the interval $(-p, p)$, the Fourier series of f converges to $f(x)$ at a point continuity. At a point of discontinuity the Fourier series converges to the average

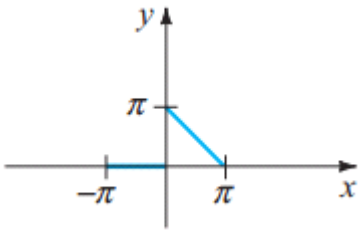
$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ are the right- and left-hand limits of f at x, respectively.

Simply Put:

When a Fourier series is continuous at a point, then it converges to that y-value

When a Fourier series is discontinuous at a point, then it convers to the average y-value



Example
Find the Fourier series of f on the given interval.

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$$

Give the number to which the Fourier series converges at a point of discontinuity of f

Example
Find the Fourier series of f on the given interval.

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$$

Give the number to which the Fourier series converges at a point of discontinuity of f

Example
Find the Fourier series of f on the given interval.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$$

Give the number to which the Fourier series converges at a point of discontinuity of f

Example
Find the Fourier series of f on the given interval.

$$f(x) = \begin{cases} 0 & -2 < x < -1 \\ -2 & -1 \le x < 0 \\ 1 & 0 \le x < 1 \\ 0 & 1 \le x < 2 \end{cases}$$

Give the number to which the Fourier series converges at a point of discontinuity of f