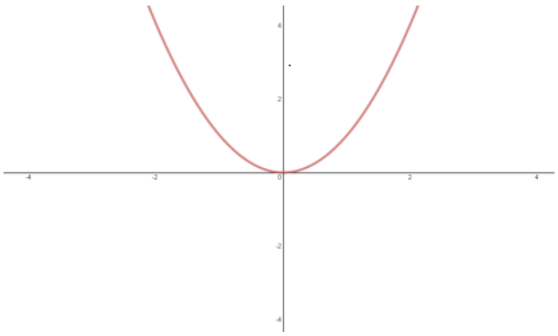
MATH 2271

Chapter 11.3

Symmetry

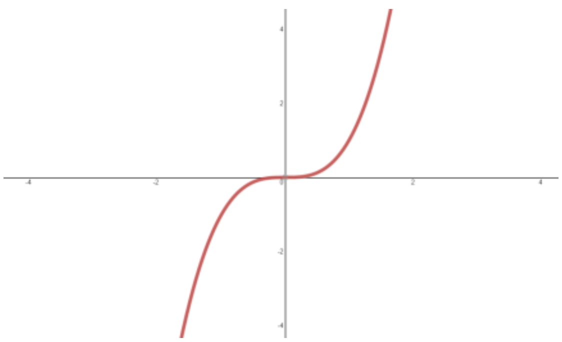
What Is Symmetry

There are 2 types of symmetry we are concerned with, odd symmetry and even symmetry. Sometimes these are referred to as odd functions or even functions.



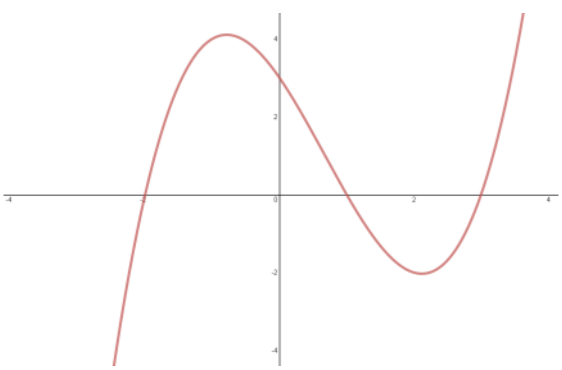
Even Symmetry
If the following is true for a function, then it shows even symmetry and it is an even function. These functions are symmetric about the y-axis

$f(-x) = f(x)$



Odd Symmetry
If the following is true for a function, then it is shows odd symmetry and it is an odd function. These functions are symmetric about the origin.

$f(-x) = -f(x)$
Or
$-f(-x) = f(x)$



No Symmetry

A function can have even symmetry OR odd symmetry OR no symmetry.

Symmetry

Example
Determine if the given function is even, odd, or neither

$$f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \le x < 1 \end{cases}$$

Recall: Fourier Series

Let p > 0 be a fixed number and f(x) be a periodic function with a period 2p, defined on (-p, p). The Fourier series of f(x) is a way of expanding the function f(x) into an infinite series involving sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right).$$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx. \qquad a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx. \qquad b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx.$$

Note: P is half the period!

From a practical point of view, it suffices to calculate the integrals above and plug them into the big formula. Couple of other things to know:

- $\sin(+-n\pi) = 0$
- $\cos(+-n\pi) = (-1)^n$
- $\cos(-n) = \cos(n)$
- $\sin(-n) = -\sin(n)$

Some useful trig identities

- $sin(a)sin(b) = ½[cos(a-b) - cos(a+b)]$
- $cos(a)cos(b) = ½[cos(a-b) + cos(a+b)]$
- $sin(a)cos(b) = ½[sin(a+b) + sin(a-b)]$
- $sin(a+b) = sin(a)cos(b) + cos(a)sin(b)$
- $cos(a+b) = cos(a)cos(b) \mp sin(a)\,sin(b)$

Fourier Series - Symmetry

The Fourier series of an even function f defined on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x,$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx.$$

The Fourier series of an odd function f defined on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x,$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx.$$

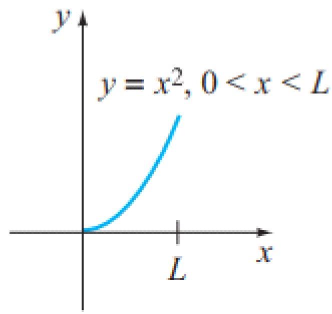
Example
Expand the given function in an appropriate cosine or sine series

$$f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$$

Fourier - Half Range

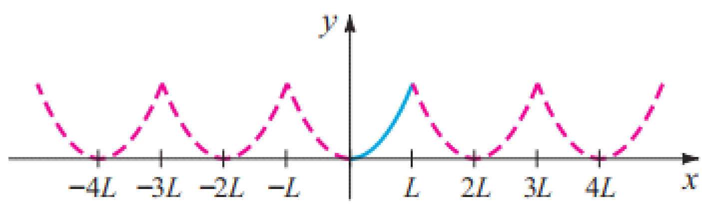
Previously, it was assumed the function was defined about the origin and the interval was (-p,p).

What if the function is only defined on some interval (0, L) instead?

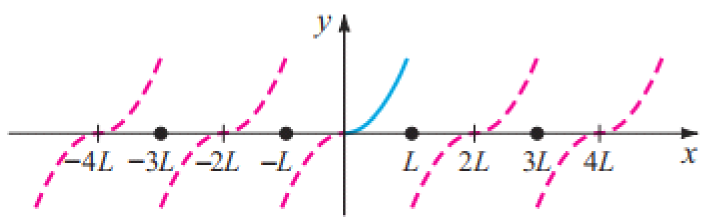


We can arbitrarily decide what the function will look like from (-L, 0).

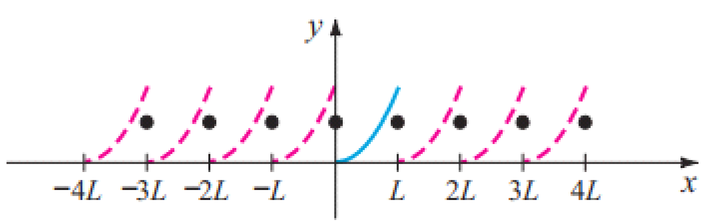
Option 1) Even Reflection, for which p=L
This is called the half range cosine series



Option 2) Odd Reflection, for which p=L
This is called the half range sine series



Option 3) Define it the same way it was defined from (0, L)
p = L/2



Example
Find the half-range cosine and sine expansion of the given function.

$$f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} \le x < 1 \end{cases}$$

Example
Find the half-range cosine and sine expansion of the given function.

$$f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$$